

## SMOOTHINGS OF NORMAL SURFACE SINGULARITIES

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THE GOAL of this paper is to study topological and analytic invariants of a smoothing of an isolated singularity, with applications to the moduli space and smoothability criteria.

The key topological object is the Milnor fibre. Let  $\{f=0\}$  be a hypersurface with isolated singular point at  $0 \in \mathbb{C}^{n+1}$ . The map (germ)  $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  is a smoothing, and the Milnor fibre  $F$  is the "localized fibre", given by  $F = B_\epsilon \cap f^{-1}(\delta)$ , for  $0 < |\delta| \leq \epsilon$ . In his book [23], Milnor proved  $F$  is a real  $2n$ -manifold with boundary, and has the homotopy type of a bouquet of  $n$ -spheres;  $\mu = rk H_n(F)$  is called the Milnor number. More generally, let  $V$  be a local analytic (or algebraic) variety with isolated singularity, reduced of pure dimension  $n$ . To any smoothing  $\pi: \mathcal{V} \rightarrow T$  of  $V$  (we always deal with germs), one can still attach a Milnor fibre  $F = B_\epsilon \cap \pi^{-1}(t)$  (where  $B_\epsilon$  is a ball in some  $\mathbb{C}^N$  containing  $\mathcal{V}$ )—see [21].  $F$  is a  $2n$ -manifold with boundary, with the homotopy type of a finite complex of dimension  $n$ . We again define  $\mu = rk H_n(F)$ ; there may be other betti numbers.

We give two analytic invariants of  $\pi: \mathcal{V} \rightarrow T$ , which measure the failure of certain objects on  $V$  to lift during deformation. Denote by  $\omega_{\mathcal{V}/T}$  the relative dualizing differentials (see 3.7); by  $\theta_{\mathcal{V}/T}$  the relative derivations; and by  $l$  the length of a finite  $\mathbb{C}$ -module. We define

$$l(\text{Coker}(\omega_{\mathcal{V}/T}^* \otimes \mathcal{O}_V \rightarrow \omega_V^*)) = \alpha$$

$$l(\text{Coker}(\theta_{\mathcal{V}/T} \otimes \mathcal{O}_V \rightarrow \theta_V)) = \beta.$$

(Note  $\alpha$  is 0 if  $V$  is Gorenstein.) The philosophy here is that  $\alpha$  and  $\beta$  depend only on  $V$  and the topology of  $F$ , and in a precise but mysterious way. We have two general conjectures.

CONJECTURE (see 1.6). *If  $V$  is normal, then  $b_1(F) = 0$ .*

In Theorem 2.2, we prove this for smoothings of negative weight for a variety with  $\mathbb{C}^*$ -action. Examples show  $\pi_1(F)$  may be infinite, and (even for  $V$  rational and Gorenstein)  $b_2(F)$  and  $b_3(F)$  may both be non-zero.

CONJECTURE (see 4.2).  *$\beta$  = dimension of the irreducible component of the moduli space of  $V$  on which the smoothing  $\pi$  occurs.*

We prove this if  $T_V^2 = 0$  (e.g.  $V$  a complete intersection), or if  $V$  is a curve, or if  $\pi$  globalizes in a certain strong sense (Theorem 4.3). This conjecture should be true in a very general deformation-theoretic setting.

The next problem is to compute  $\alpha$ ,  $\beta$ , and the topological Euler characteristic  $\chi_T(F)$  in terms of invariants of  $V$ , especially in the case  $V$  a curve or surface. We have in mind the following known theorems.

(1) (Milnor [23]). If  $V$  is a curve, then  $\mu = 2\delta - r + 1$ , where  $r$  = number of branches,  $\delta$  = "number of double points".

(2) (Deligne[7]). If  $V$  is a curve, then any smoothing component of the moduli space has dimension  $3\delta - l(\theta_{\tilde{V}}/\theta_V)$ , where  $\tilde{V}$  = normalization of  $V$ .

(3) (Durfée[9], Laufer[20]). If  $V$  is a hypersurface in  $\mathbb{C}^3$ , then  $\mu$  and the signature  $\sigma(F)$  may be computed from resolution invariants (for the formulas, see, 3.15).

Thus, for a curve  $\chi_T(F)$  and the dimension of a smoothing component ( $= \beta$ ) are independent of the smoothing; and Laufer's method suggests that for a Gorenstein surface,  $\chi_T(F)$  is independent. On the other hand, consider the cone over the rational quartic curve in  $\mathbb{P}^4$ ; there are smoothing components of dimensions 3 and 1, giving Milnor fibres with  $\mu = 1$  and 0, respectively. Our results suggest that for a normal surface  $\chi_T(F) - \alpha$  and  $\beta - 2\alpha$  depend only on  $V$ , not on the smoothing. So, the smoothing components of a Gorenstein surface should have the same dimension.

The formulas we give are valid if the smoothing  $\pi: \mathcal{V} \rightarrow T$  globalizes in the following sense (see 3.1.1):

(G) There is a compact analytic space  $Y$  containing  $V$  and non-singular elsewhere, and a (proper) smoothing  $\mathcal{Y} \rightarrow T$  inducing  $\pi$ .

This assumption allows one to use the Riemann–Roch Theorem to compare cohomology on  $Y$ ,  $Y$ , and a resolution  $\tilde{X}$  of  $Y$ ; our method is a variant of the one in [7, 20, 23], and yields more or less uniform proofs of results (1)–(3) above. We prove in Theorem 3.3 that (G) is valid if either:  $V$  is a complete intersection;  $V$  is a two-dimensional cusp for the Hilbert modular group[14]; the smoothing is determinantal or Pfaffian;  $V$  has  $\mathbb{C}^*$ -action and  $\pi$  has negative weight.

**THEOREM (see 3.13).** *Let  $\pi: \mathcal{V} \rightarrow T$  be a smoothing of the normal surface singularity  $V$ , with Milnor fibre  $F$ . Let  $X \rightarrow V$  be the minimal resolution, with exceptional fibre  $E$ , consisting of  $r$  curves. If (G) is satisfied, then*

- (a)  $\chi_T(F) = 13h^1(\mathcal{O}_X) + \chi_T(E) - h^1(\omega_X^*) + \alpha$
- (b)  $\sigma(F) = 4h^1(\mathcal{O}_X) - (\chi_T(F) - \chi_T(E)) - r$
- (c)  $\beta = h^1(\theta_X) - 14h^1(\mathcal{O}_X) + 2(\chi_T(F) - \chi_T(E))$ .

**COROLLARY.** *Let  $V$  be a complete intersection or Gorenstein of codimension 3, and write  $\omega_X = \mathcal{O}_X(-Z)$ , some  $Z$ . Then*

- (a) (Laufer)  $\chi_T(F) = 12h^1(\mathcal{O}_X) + \chi_T(E) + Z \cdot Z$
- (b) (Durfée)  $\sigma(F) = -8h^1(\mathcal{O}_X) - Z \cdot Z - r$
- (c)  $\dim T_V^1 = \dim (\text{versal deformation}) = h^1(\theta_X) + 10h^1(\mathcal{O}_X) + 2Z \cdot Z$ .

**COROLLARY.** *Let  $V = \mathbb{C}^2/G$  be a quotient singularity. Then*

- (a) *the Artin component has maximal dimension among the components of the moduli space.*
- (b) *the difference in dimensions of two components is even.*

(These results are in §3 and §4.) We also prove (Theorem 3.20) that for an Artin component smoothing of a rational singularity,  $\alpha$  is “as large as possible”; in particular, it is never 0 once  $\text{mult } V \geq 4$ .

Now suppose  $V$  has a  $\mathbb{C}^*$ -action. Greuel has proved[11] that if  $V$  is a complete intersection (of positive dimension), then  $\mu = \dim T_V^1$ . We conjecture (4.7) that for a Gorenstein surface with  $\mathbb{C}^*$ -action and a smoothing  $\pi: \mathcal{V} \rightarrow T$ ,  $\mu = \dim (\text{smoothing})$

component of  $\pi$ ). We prove this (Theorem 4.10) assuming globalization and the conjecture about  $\beta$ ; hence we give another proof of Greuel's complete intersection result in dimension 2. This proof is hard, and uses Hochster's work on the Zariski-Lipman conjecture.

In [20], Laufer suggested that a criterion for smoothability for a Gorenstein surface might be

$$12h^1(\mathcal{O}_X) + \chi_T(E) + Z \cdot Z \geq 0,$$

i.e. (assuming the formula)  $\chi_T(F) \geq 0$ . In fact, the "correct" version should be

CONJECTURE (5.2). *For a smoothable Gorenstein surface,*

$$10h^1(\mathcal{O}_X) + \chi_T(E) + Z \cdot Z \geq 1,$$

i.e.  $\chi_T(F) \geq 2h^1(\mathcal{O}_X) + 1$ .

The point is that  $b_1(F)$  should be 0, and  $\mu_0 + \mu_+ = 2h^1(\mathcal{O}_X)$  (where we consider the intersection pairing on  $H_2(F)$ ). We prove the conjecture for smoothings of negative weight, thus recovering a theorem of Pinkham[27] concerning negative smoothability of a special class of singularities (5.5). We can also prove the result for a cusp.

THEOREM (5.6). *Let  $V$  be a two-dimensional cusp, of multiplicity  $m$ , and with  $r$  exceptional curves in the minimal resolution. If  $V$  is smoothable, then*

$$m \leq r + 9.$$

We give examples of smoothable cusps with  $m = r + 9$ , for arbitrarily large  $m$  (5.9.4), by taking cyclic quotients of hypersurfaces; we had previously proved[35]  $V$  is smoothable if  $r > m^2 - m$ . The smoothable cusps with  $m = r + 9$  give counterexamples to a conjecture of Durfee[9] that the signature of  $F$  is  $\leq 0$ .

The quotient construction mentioned above also gives examples (4.9.2) of smoothings of rational singularities for which  $\mu = 0$ ; thus,  $F$  is a rational homology ball. The boundary of  $F$  (=link of  $V$ ) is thus a compact 3-manifold bounding a rational homology ball. We construct a triply-infinite family of such links; this is (part of) a triply-infinite family recently discovered (independently) by Neumann[25], using differential topology.

The paper is organized as follows: §1 gives the basics on the Milnor fibre. The case of smoothings of negative weight is considered in §2, especially plethora of interesting examples arising from projective non-singular varieties via taking cones. We derive formulas for the invariants in §3, assuming globalization hypotheses, while §4 considers dimensions of smoothing components. Some examples are constructed in §5, and smoothability criteria are discussed. In §6 we prove that certain kinds of smoothings can be globalized. The appendix gives a simple but useful way to compute how "duals fail to lift under deformation", important in finding bounds for  $\alpha$  and  $\beta$ .

## §1. GENERALITIES ON THE MILNOR FIBRE

(1.1) Let  $V$  be a reduced complex algebraic variety (or local analytic space) of dimension  $n$ , with isolated singularity at  $o$ . Denote by  $R$  the local ring of  $V$  at  $o$ . A smoothing is a flat morphism  $\pi : \mathcal{V} \rightarrow T$  of local spaces, plus an isomorphism  $\pi^{-1}(o) \xrightarrow{\sim} V$ , such that  $\pi^{-1}(t) = V_t$  is non-singular for  $t \neq 0$ ; we may as well assume  $T$  is a disk

or the spectrum of a discrete valuation ring. Choose a local embedding  $V \subset \mathbb{C}^N$ , and a small  $\epsilon$ -ball  $B_\epsilon$  so that  $\mathcal{V}$  intersects  $S_\epsilon = \partial B_\epsilon$  transversally; then  $B_\epsilon \cap V_t = F$  is independent (up to isotopy) of small  $\epsilon$  and  $t \neq 0$ , and of the embedding of  $\mathcal{V}$  ([21, 23]). We call  $F$  the Milnor fibre of the smoothing; it is a real compact  $2n$ -manifold with boundary  $\partial F = K$ , the link of  $V$ . Note  $F$  is diffeomorphic to  $F - \partial F$ .

(1.2) If  $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$  is a polynomial with isolated critical point at the origin, then  $f$  is a smoothing of the hypersurface  $V = \{f = 0\}$ . Milnor proved [23] that  $F$  has the homotopy type of a bouquet of  $n$ -spheres,  $\mu$  in number; the module  $H_n(F; \mathbb{Z}) = \mathbb{Z}^\mu$  contains the vanishing cycles. In particular, the betti numbers  $b_i(F)$  equal 0,  $i \neq 0, n$ . Hamm [12] proved the analogous result for smoothings of a complete intersection.

(1.3) In the setting of (1.1),  $F$  always has the homotopy type of a finite CW-complex of dimension  $\leq n$ . This follows by the Andreotti–Frankel proof of the Lefschetz theorems using Morse theory ([1]; see also [23], §5). In particular, all cohomology vanishes past dimension  $n$ . For an algebraic proof of the vanishing of (étale) cohomology in this range use SGA VII, Exposé I [6].

(1.4) From now on, consider only real homology and cohomology. Lefschetz duality asserts that cup product gives a perfect pairing

$$H^i(F) \times H^{2n-i}(F, \partial F) \rightarrow H^{2n}(F, \partial F) \cong \mathbb{R}.$$

Therefore,

$$H^{2n-i}(F, \partial F) \cong H_i(F).$$

By (1.3),

$$H^i(F) = H_i(F) = 0, \quad i > n.$$

$$H^i(F, \partial F) = H_i(F, \partial F) = 0, \quad i < n.$$

The long exact sequence in cohomology then yields

$$b_i(F) = b_i(\partial F), \quad i \leq n-2$$

$$b_{n-1}(F) \leq b_{n-1}(\partial F).$$

Let us define finally

$$\mu = b_n(F).$$

(1.5) If  $n$  is even, the intersection pairing on  $F$  is a symmetric bilinear form on  $H_n(F)$  defined via

$$H^n(F, \partial F) \times H^n(F, \partial F) \rightarrow H^n(F) \times H^n(F, \partial F) \rightarrow H^{2n}(F, \partial F).$$

Let  $\mu_0$ ,  $\mu_+$ , and  $\mu_-$  denote the number of 0's, +1's, and -1's in the real diagonalization of the form. We denote the signature by  $\sigma = \mu_+ - \mu_-$ . It follows from the long

exact sequence in cohomology that

$$(1.5.1) \quad \mu_0 = b_{n-1}(\partial F) - b_{n-1}(F).$$

We offer the

CONJECTURE 1.6. *If  $V$  is normal, then  $b_1(F) = 0$ .*

*Remarks (1.7.1).* Aside from the complete intersection case mentioned earlier, the conjecture is true in the following cases:

- (a)  $b_1(\partial F) = 0$  (1.4)
- (b)  $V$  has a  $C^*$ -action, and the smoothing has negative weight (2.2, below)
- (c)  $V$  is a two-dimensional cusp (E. Looijenga—to appear).

On the other hand,  $\pi_1(F)$  may be infinite (2.6.1).

(1.7.2) In (2.7), we give a Gorenstein, rational, 3-dimensional singularity in  $C^7$  with  $b_2(F) = 2$ ,  $b_3(F) = 1$ . Presumably, one needs depth conditions on the  $\Omega_V^i$  to get more vanishing betti numbers of  $F$ .

## §2. SINGULARITIES WITH $C^*$ -ACTION

(2.1)  $V$  is called weighted homogeneous if it has a good  $C^*$ -action; equivalently,  $V = \text{Spec } A$ , where  $A$  is a (positively) graded ring. So,  $V$  is defined as an affine variety by weighted homogeneous polynomials  $\{f_i\}$ . We assume  $V$  normal, of dimension  $\geq 2$ . There is a natural compactification  $\bar{V} = \text{Proj } A[t]$ , where  $t$  has weight 1;  $\bar{V} - V = \bar{E} = \text{Proj } A$  is a Weil divisor “at  $\infty$ ”, and  $\bar{V}$  and  $\bar{E}$  are normal, with only cyclic quotient singularities (see [27], §2, for this and the following). A smoothing  $\mathcal{V} \rightarrow T$  has negative weight if  $\mathcal{V}$  is defined by adding to each  $f_i$  terms of weight no bigger than that of  $f_i$ . Such a smoothing induces a projective deformation  $\bar{\mathcal{V}} \rightarrow T$  of  $\bar{V}$ . Since  $A$  is normal, then  $\bar{\mathcal{V}} \rightarrow T$  is locally trivial near  $\bar{E}$ , so the singularities can be simultaneously resolved. This gives a projective deformation  $\mathcal{Y} \rightarrow T$  of  $Y$ , a desingularization of  $\bar{V}$  along  $\bar{E}$ ; thus  $V \subset Y$ ,  $Y - V$  is a union of non-singular divisors, intersecting transversally, and  $Y$  is non-singular along  $Y - V$ . Note these smoothings give deformations of the whole affine variety  $V$ .

THEOREM 2.2. *Let  $\mathcal{V} \rightarrow T$  be a smoothing of negative weight of a normal singularity  $V$  with good  $C^*$ -action, and denote by  $V_t$  the general affine fibre. Then*

- (a)  $V_t$  is diffeomorphic to the Milnor fibre  $F$
- (b)  $b_1(F) = 0$ .

*Proof.* First, we show  $V_t \approx F$ , where  $\approx$  means diffeomorphic. We may assume  $T$  is a disk  $D$ . Write

$$\mathcal{V} \subset C^N \times D \subset C^{N+1}.$$

Now, for any  $\epsilon > 0$ ,

$$B_\epsilon(C^{N+1}) \cap V = B_\epsilon(C^N) \cap V.$$

Next, we claim for any  $\epsilon, \epsilon' > 0$ ,

$$(2.2.1) \quad B_\epsilon(\mathbb{C}^N) \cap V \approx B_{\epsilon'}(\mathbb{C}^N) \cap V \approx V.$$

This follows from Morse theory once we show that  $\sum_1^N \|z_i\|^2$  on  $\mathbb{C}^N$  has no critical points on  $V$  (except at  $0 \in V$ ). For,  $V$  is non-singular away from  $0$ ; so, we show that if  $Q \in V$ ,  $Q \neq 0$ , then direction  $\vec{0Q}$  is not normal to  $V$  at  $Q$ . But the  $\mathbb{C}^*$ -action on  $V$  gives that if  $a_1, \dots, a_N$  are the weights of the variables, then  $(a_1 z_1, \dots, a_N z_N)$  is a tangent vector to  $V$  at  $Q = (z_1, \dots, z_N)$ . Taking inner products (Euclidean on  $\mathbb{R}^{2N} =$  hermitian on  $\mathbb{C}^N$ ) gives  $\sum a_i \|z_i\|^2 \neq 0$ , whence the claim.

It follows from (2.2.1) that for  $t \in D$ ,  $0 < \|t\| \leq \min(\epsilon, \epsilon')$ .

$$(2.2.2) \quad B_\epsilon(\mathbb{C}^{N+1}) \cap V_t \approx B_{\epsilon'}(\mathbb{C}^{N+1}) \cap V_t \approx V_t.$$

(This proves  $V_t \approx F$ .) For, with  $\epsilon' < \epsilon$  and  $t$  small,

$$(B_\epsilon - B_{\epsilon'}) \cap V_t \approx (B_\epsilon - B_{\epsilon'}) \cap V.$$

Next, complete the family to a deformation  $\mathcal{Y} \rightarrow T$  of  $Y$ , as above. Then  $Y_t - V_t$  is a divisor  $C_t$ , which is topologically the same for all  $t$ . We assert that

$$(2.2.3) \quad H_1(Y_t) = 0,$$

itself a consequence of

$$(2.2.4) \quad H^1(\mathcal{O}_{\bar{V}}) = 0.$$

For, (2.2.4) implies  $H^1(\mathcal{O}_Y) = 0$  (by the Leray spectral sequence of  $Y \rightarrow \bar{V}$ ), hence  $H^1(\mathcal{O}_{Y_t}) = 0$  (semi-continuity), hence  $H^1(Y_t) = 0$  (Hodge theory).

Now,  $\bar{V} = \text{Proj } A[t]$ .  $A$ , being normal, has depth  $\geq 2$ , so  $A[t]$  has depth  $\geq 3$ , whence

$$H_P^i(\text{Spec } A[t], \mathcal{O}) = 0, i \leq 2$$

( $P$  = vertex of cone). Thus, writing  $U = \text{Spec } A[t] - \{P\}$ ,

$$H^1(U, \mathcal{O}_U) = 0.$$

This implies  $H^1(\bar{V}, \mathcal{O}_{\bar{V}}(n)) = 0$ , all  $n$  (see [8], 1.1.4), hence (2.2.4). (Pinkham proved (2.2.4) for surfaces in [27].) This proves (2.2.3).

Finally, we must prove

$$(2.2.5) \quad H_1(V_t) = 0.$$

By Lefschetz duality,

$$H_1(V_t) \simeq H^{2n-1}(Y_t, C_t).$$

By (2.2.3) and duality,  $H^{2n-1}(Y_t) = 0$ , whence we have the exact cohomology sequence

$$H^{2n-2}(Y_t) \xrightarrow{\alpha} H^{2n-2}(C_t) \rightarrow H^{2n-1}(Y_t, C_t) \rightarrow 0.$$

Since  $H^{2n-2}(C_t) \cong \bigoplus H^{2n-2}(C_{t,i})$ , where  $C_t$  is the union of smooth irreducible divisors  $C_{t,i}$ , it is not hard to see that the surjectivity of  $\alpha$  is equivalent to the independence of the classes  $[C_{t,i}] \in H^2(Y_t)$ . But homological and algebraic equivalence coincide for divisors (e.g. [10], p. 462); and  $H^1(\mathcal{O}_{Y_t}) = 0$ . (2.23), algebraic equivalence is linear equivalence mod torsion. Therefore, a homological equivalence relation on the  $C_{t,i}$ 's would give a non-trivial linear equivalence relation on them; thus, there is a non-constant rational function  $f$  on  $Y_t$  whose zero and polar divisors are contained in  $C_t$ . Since  $Y_t \rightarrow \bar{V}_t$  is birational,  $f$  is a function on  $\bar{V}_t$  with zero and polar divisors contained in the irreducible Weil divisor  $\bar{E}_t$ . This is impossible if  $f$  is not constant ( $V_t$  is normal). Therefore, the map  $\alpha$  is surjective, whence (2.2.5) and the Theorem.

(2.3) A standard example of a normal  $V$  with  $C^*$ -action is a cone  $V = \text{Spec} \bigoplus \Gamma(C, L^{\otimes m})$ , where  $C$  is a projective nonsingular variety of dimension  $n-1$ , and  $L$  is an ample line bundle. Since  $V$  is constructed by blowing down the 0-section of the (geometric) line bundle  $L^* \rightarrow C$ , one concludes  $V$  is normal, and the depth of  $V$  at the vertex satisfies

$$\text{depth } V \geq i+2 \text{ iff } H^i(C, L^{\otimes m}) = 0, 1 \leq i \leq n, \text{ all } m$$

(compute local cohomology as cohomology on  $L^* - \{0\text{-section}\}$ ). Note that the link  $K$  of  $V$  is the circle bundle arising from  $L^* \rightarrow C$ . Finally, if  $L$  defines a projectively normal embedding  $C \subset \mathbb{P}^N$ , then  $V$  is the usual cone.

(2.4) Suppose  $X$  is projective and non-singular, of dimension  $n$ . Assume  $C \subset X$  is an ample, non-singular divisor, with

$$H^1(\mathcal{O}_X(mC)) = 0, \text{ all } m.$$

Let  $\mathcal{V} = \text{Spec} \bigoplus_m \Gamma(X, \mathcal{O}_X(mC))$ ,  $t \in \Gamma(X, \mathcal{O}_X(C))$  the section defining  $C$ , and  $\pi: \mathcal{V} \rightarrow \text{Spec } \mathbb{C}[t]$  the natural map. Then  $\pi$  is a smoothing of negative weight of  $V = \text{cone over } C$  (see [26], p. 46). Geometrically, assuming  $\mathcal{O}_X(C)$  is very ample and projectively normal,  $\pi$  is constructed by moving a hyperplane through the vertex of the cone  $\mathcal{V}$ . We give some examples based on this construction.

**PROPOSITION 2.5.** *Let  $X$  be a projective non-singular surface with  $b_1(X) = 0$ , and  $C$  an ample non-singular curve on  $X$ , with  $h^1(\mathcal{O}_X(mC)) = 0$ , all  $m$ . As above this gives a smoothing of  $V = \text{Spec} \bigoplus \Gamma(C, \mathcal{O}_C(mC))$ , with Milnor fibre  $F = X - C$ . Then*

$$(a) \quad b_1(F) = 0$$

$$(b) \quad \pi_1(F) \rightarrow \pi_1(X) \text{ is surjective}$$

$$(c) \quad \mu_+ = 2p_g$$

$$\mu_- = h^{1,1} - 1$$

$$\mu_0 = 2 \text{ genus } (C).$$

$$(d) \sigma(F) = \sigma(X) - 1.$$

*Proof.* (a) is in (2.2), and (b) is obvious via transversality. Next, it is well known that the intersection pairing on  $X$  is non-degenerate, of type  $(2p_g + 1, h^{1,1} - 1)$ . Using the exact sequence

$$0 \rightarrow H^1(C) \rightarrow H^2(X, C) \rightarrow H^2(X) \rightarrow H^2(C) \rightarrow 0$$

and the definition of the intersection pairing on  $H_2(F)$ , one concludes (c) and (d).

*Remarks (2.6.1).* IF  $X$  is Mumford's surface of general type with  $p_g = q = 0$ ,  $h^{1,1} = 1$  ([24]), then  $\pi_1(X)$  is infinite, hence (by, b) so is  $\pi_1(F)$ . There exists similar examples using elliptic surfaces.

(2.6.2) Durfee has conjectured ([9], 5.2) that  $\sigma(F) \leq 0$  for any smoothing fibre in dimension 2. By the Proposition, this conjecture would imply there are no projective smooth surfaces with  $b_1(X) = 0$ ,  $\sigma(X) > 1$ ; indeed, checking the known surfaces with positive index (Kodaira surfaces and quotients of the open two-ball in  $\mathbb{C}^2$ ), we find no such example. However, we give a different counterexample to Durfee's conjecture in (5.9.3) below.

(2.6.3) Using the work of Pinkham [27], one can prove a stronger result than (2.5) for surfaces with  $\mathbb{C}^*$ -actions (and not just cones).

*Example 2.7.* There is a 3-dimensional rational, Gorenstein,  $\mathbb{C}^*$ -action singularity with a Milnor fibre  $F$  for which  $b_2(F) = 2$ ,  $b_3(F) = 1$ .

*Proof.* Let  $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ,  $L = \mathcal{O}_1(1) \otimes \mathcal{O}_2(1) \otimes \mathcal{O}_3(1)$  the natural very ample line bundle. Let  $S \subset X$  be a non-singular surface with  $\mathcal{O}_X(S) \simeq L$ . It is not hard to show that

- (i)  $S$  is  $\mathbb{P}^1 \times \mathbb{P}^1$  blown-up at 2 distinct points of the diagonal
- (ii)  $\mathcal{O}_S(L) \simeq K_S^{-1}$ , and is very ample.

Using the construction of (3.4), let  $t \in \Gamma(X, L)$  define  $S$ ; then

$$\text{Spec } \bigoplus_{m=0}^{\infty} \Gamma(X, L^{\otimes m}) \rightarrow \text{Spec } \mathbb{C}[t]$$

gives a smoothing of

$$\text{Spec } \bigoplus_{m=0}^{\infty} \Gamma(S, L^{\otimes m}|_S) = \text{Spec } R$$

with Milnor fibre  $F = X - S$ .

By (ii), and some straightforward vanishing results, one deduces  $R$  is Gorenstein and rational. To compute  $H_i(X - S) = H^{6-i}(X, S)$ , use the cohomology sequence

$$H^*(X, S) \rightarrow H^*(X) \rightarrow H^*(S).$$



Note  $H^2(X, S) = H^3(X, S) = 0$ , by (1.4) and (2.2.b), respectively. Standard computations yield

$$b_2(X) = 3, b_3(X) = 0, b_4(X) = 3$$

$$b_2(S) = 4, b_3(S) = 0, b_4(S) = 1.$$

The result follows.

### §3. FORMULAS FOR $\chi_T(F)$ , $\sigma(F)$ AND $\beta$

(3.1) If  $\pi: \mathcal{V} \rightarrow T$  is a smoothing of  $V$  with Milnor fibre  $F$ , we seek formulas for the topological Euler characteristic  $\chi_T(F)$  (and, if  $\dim V$  is even, the signature  $\sigma(F)$ ) in terms of invariants of some resolution  $f: X \rightarrow V$ . The method below is a strengthening of earlier work ([7, 20, 23]), and applied whenever the family  $\pi$  globalizes in the sense of:

(3.1.1) There is a compact analytic space  $Y$ , with one singular point  $P$ , at which  $Y$  is locally isomorphic to  $V$ ; and a proper smoothing  $\mathcal{Y} \rightarrow T$  of  $Y$  which induces  $\mathcal{V} \rightarrow T$ .

(3.2) That is,  $V$  can be “completed” to a compact  $Y$  with no other singularities, and the deformation of  $V$  can be realized as a deformation of  $Y$ . If  $H^2(\theta_Y) = 0$ , this second condition is automatic, since by obstruction theory local deformations can be patched globally (we give a proof of this more-or-less well-known result in 6.4 below). We know of no examples where (3.1.1) fails, but it is generally hard to produce  $\mathcal{Y}$  from  $\mathcal{V}$ . We do have

**THEOREM 3.3.** *The globalization (3.1.1) is valid if  $V$  is either*

(a) *a curve; (b) a complete intersection; (c) a two-dimensional cusp for the Hilbert modular group [14]; (d) a singularity with  $\mathbb{C}^*$ -action, and the smoothing has negative weight; (e) determinantal, and the smoothing is determinantal ( $\dim V < 4$ ); (f) Pfaffian, and the smoothing is Pfaffian ( $\dim V < 7$ ) (see 6.3).*

*Proof.* (a) follows because a curve can be completed, and  $H^2(\theta_Y) = 0$  automatically. (b) is well-known (or, use the argument of 6.2 below), and (d) was mentioned in (2.1). The other proofs are postponed to §6.

(3.4) If  $V$  is Cohen–Macaulay of codimension 2, then by a theorem of Hilbert,  $V$  is defined by the  $r \times r$  minors of an  $r \times (r+1)$  matrix; all deformations of  $V$  are determinantal, and  $V$  is smoothable if  $\dim V < 4$  [30]. If  $V$  is Gorenstein of codimension 3, then by a theorem of Buchsbaum–Eisenbud,  $V$  is defined by the  $2r \times 2r$  Pfaffians of a skewsymmetric  $(2r+1) \times (2r+1)$  matrix; all deformations of  $V$  are Pfaffian, and  $V$  is smoothable if  $\dim V < 7$  [3]. Thus, these cases are covered by Theorem 3.3.

(3.5) Assume the globalization (3.1.1). Mayer–Vietoris gives for  $t \neq 0$

$$(3.5.1) \quad \chi_T(Y_t) - \chi_T(Y) = \chi_T(F) - 1,$$

since  $\mathcal{Y} \rightarrow T$  is locally trivial away from the singularity of  $Y$ ; write, e.g.

$$Y_t = (Y_t \cap B_\epsilon) \cup (Y_t - (Y_t \cap B_\epsilon))^0.$$

Second, if  $D$  is a small neighborhood of  $P$  on  $Y$  and  $f: \bar{X} \rightarrow Y$  is a desingularization, then

$$(3.5.2) \quad \chi_T(\bar{X}) - \chi_T(Y) = \chi_T(f^{-1}(D)) - 1.$$

Thus,

$$(3.5.3) \quad \chi_T(F) = \chi_T(Y_t) - \chi_T(\bar{X}) + \chi_T(f^{-1}(D)).$$

Similarly, Novikov additivity for signatures gives ( $\dim V$  even)

$$(3.5.4) \quad \sigma(F) = \sigma(Y_t) - \sigma(\bar{X}) + \sigma(f^{-1}(D)).$$

**THEOREM 3.6** (Milnor[23]). *For a smoothable curve singularity,*

$$\mu = 2\delta - r + 1,$$

where  $\delta$  = "number of double points," and  $r$  = number of branches.

*Proof.* Globalize as before (3.3.(a)); the resolution  $\bar{X} \rightarrow Y$  is the normalization  $f: \bar{Y} \rightarrow Y$ . By definition,

$$\delta = l(\text{Coker}(\mathcal{O}_Y \rightarrow f_*\mathcal{O}_{\bar{Y}})).$$

Therefore,

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_{\bar{Y}}) - \delta.$$

$\bar{Y}$  is the disjoint union of non-singular curves. By Riemann–Roch,

$$\chi(\mathcal{O}_{\bar{Y}}) = \frac{1}{2} \chi_T(\bar{Y})$$

$$\chi(\mathcal{O}_{Y_t}) = \frac{1}{2} \chi_T(Y_t) \quad (t \neq 0).$$

Also,  $f^{-1}(D)$  is the disjoint union of  $r$  contractible curves, so

$$\chi_T(f^{-1}(D)) = r.$$

Finally, by semi-continuity,  $\chi(\mathcal{O}_{Y_t}) = \chi(\mathcal{O}_Y)$ . But  $\mu = 1 - \chi_T(F)$ ; plugging the above formulas into (3.5.3) gives the result.

(3.7) Assume for the rest of §3 that  $V$  has a normal surface singularity, with local ring  $R = \mathcal{O}_{V,0}$ . For a family  $\mathcal{Y} \rightarrow T$ , recall one has the sheaf of relative dualizing differentials  $\omega_{\mathcal{Y}/T}$  ([22, 35]); it is flat over  $T$ , and induces  $\omega_{Y_t}$  on each fibre. (If  $V$  is Gorenstein,  $\omega_{\mathcal{Y}/T}$  is invertible.) The dual  $\omega_{\mathcal{Y}/T}^*$  is still flat over  $T$ , and induces  $\omega_{Y_t}^*$  for  $t \neq 0$ ; but in general, the inclusion

$$\omega_{\mathcal{Y}/T}^* \otimes \mathcal{O}_Y \subset \omega_Y^*$$

has a cokernel of length  $\alpha \leq l(\text{Ext}_R^1(\omega, R))$ . (Collary A.2 of the appendix.) Recall the smoothing is said to be  $\omega^*$ -constant iff  $\alpha = 0$  (automatic if  $V$  Gorenstein). By semi-continuity, for  $t \neq 0$ ,

$$\chi(\omega_{Y_t}^*) = \chi(\omega_Y^*) - \alpha.$$

(3.8) Although  $\Omega_{Y/T}^1$  need not be  $T$ -flat, its dual  $\theta_{Y/T}$  certainly is, and induces  $\theta_Y$ , for  $t \neq 0$ . As before, using Corollary A.2, the inclusion

$$\theta_{Y/T} \otimes \mathcal{O}_Y \subset \theta_Y$$

will have a cokernel of length  $\beta \leq l(\text{Ext}_R^1(\Omega_R^1, R)) = l(T_R^1)$ . See §4 for more discussion of  $\beta$ ; in particular, we show  $l(T_R^2) = 0$  implies  $\beta = l(T_R^1)$ .

Again, for  $t \neq 0$ ,

$$\chi(\theta_{Y_t}) = \chi(\theta_Y) - \beta.$$

(3.9) With  $f: \bar{X} \rightarrow Y$  the minimal resolution, one has

$$(3.9.1) \quad \mathcal{O}_Y \xrightarrow{\sim} f_*(\mathcal{O}_{\bar{X}})$$

$$(3.9.2) \quad f_*\omega_{\bar{X}}^* \xrightarrow{\sim} \omega_Y^*$$

$$(3.9.3) \quad f_*\theta_{\bar{X}} \xrightarrow{\sim} \theta_Y.$$

The second assertion follows from the “easy vanishing theorem”  $H_E^1(\omega_{\bar{X}}^*) = 0$ , where  $E = f^{-1}(P) \subset \bar{X}$ ; see [35], 2.13, for a proof where  $V$  has a rational singularity, the general case being the same. (3.9.1) and (3.9.3) are standard.

(3.10) The Leray spectral sequence for  $f: \bar{X} \rightarrow Y$  yields (for coherent  $\mathcal{F}$  on  $\bar{X}$ )

$$\chi(\mathcal{F}) = \chi(f_*\mathcal{F}) - h^0(R^1f_*\mathcal{F}).$$

If  $\mathcal{F} = \mathcal{O}_{\bar{X}}$ ,  $\theta_{\bar{X}}$ , etc. we will write  $h^1(\mathcal{O}_{\bar{X}})$ ,  $h^1(\theta_{\bar{X}})$ , etc. for  $h^0(R^1f_*\mathcal{F})$ , as these integers depend only on  $R$  (and its minimal resolution  $X \rightarrow V$ ). Since  $\chi(\mathcal{O}_{Y_t}) = \chi(\mathcal{O}_Y)$ , (3.7)–(3.9) yield

$$(3.10.1) \quad \chi(\mathcal{O}_{Y_t}) - \chi(\mathcal{O}_{\bar{X}}) = h^1(\mathcal{O}_{\bar{X}})$$

$$(3.10.2) \quad \chi(\omega_{Y_t}^*) - \chi(\omega_{\bar{X}}^*) = h^1(\omega_{\bar{X}}^*) - \alpha$$

$$(3.10.3) \quad \chi(\theta_{Y_t}) - \chi(\theta_{\bar{X}}) = h^1(\theta_{\bar{X}}) - \beta.$$

(3.11) Recalling (3.5), the exceptional fibre  $E$  is a deformation retract of  $f^{-1}(D)$ , and the intersection pairing  $(E_i \cdot E_j)$  is negative definite. So, if  $r = rk H_2(E) = \text{number of exceptional curves}$ , we have

$$(3.11.1) \quad \chi_T(Y_t) - \chi_T(\bar{X}) = \chi_T(F) - \chi_T(E)$$

$$(3.11.2) \quad \sigma(Y_t) - \sigma(\bar{X}) = \sigma(F) + r.$$

(3.12) According to Riemann–Roch and the Hodge index theorem for a non-singular complex analytic surface  $Z$  ([13, 17]) (note  $\omega_Z^* = K_Z^*$ ),

$$\chi(\omega_Z^*) = 13\chi(\mathcal{O}_Z) - \chi_T(Z)$$

$$\chi(\theta_Z) = 14\chi(\mathcal{O}_Z) - 2\chi_T(Z)$$

$$\sigma(Z) = 4\chi(\mathcal{O}_Z) - \chi_T(Z).$$

Putting everything together, we easily deduce

**THEOREM 3.13.** *Let  $\pi: \mathcal{V} \rightarrow T$  be a smoothing of the normal surface singularity  $V$ , with Milnor fibre  $F$ . Let  $X \rightarrow V$  be the minimal resolution, with exceptional fibre  $E$ , consisting of  $r$  curves. If the smoothing globalizes as in (3.1.1), then*

(a)  $\chi_T(F) = 13 h^1(\mathcal{O}_X) + \chi_T(E) - h^1(\omega_X^*) + \alpha$ , where  $0 \leq \alpha \leq l(\text{Ext}_R^1(\omega, R))$ , and  $\alpha = 0$  iff  $\pi$  is  $\omega^*$ -constant

(b)  $\sigma(F) = 4h^1(\mathcal{O}_X) - (\chi_T(F) - \chi_T(E)) - r$

(c)  $\beta = h^1(\theta_X) - 14h^1(\mathcal{O}_X) + 2(\chi_T(F) - \chi_T(E))$ , where  $\beta = l(\text{Coker}(\theta_{\mathcal{V}/T} \otimes \mathcal{O}_V \subset \theta_V))$ .

(d)  $\mu_0 + \mu_+ = 2h^1(\mathcal{O}_X)$  (see 1.5).

*Proof.* Only (d) requires proof. By (1.5.1),  $\mu_0 = b_1(\partial F) - b_1(F) = b_1(E) - b_1(F)$  (see [9], p. 92). So,

$$\chi_T(F) = 1 - b_1(E) + \mu_+ + \mu_- + 2\mu_0$$

$$\sigma(F) + \chi_T(F) = 1 - b_1(E) + 2(\mu_0 + \mu_+).$$

Computing from (b), using  $\chi_T(E) = 1 - b_1(E) + r$ , gives the result.

**Remarks (3.14.1).** The whole point of (c) is Conjecture 4.2 below, which asserts  $\beta$  is the dimension of the irreducible component of the moduli space of  $V$  on which the smoothing  $\pi$  lies.

(3.14.2). The formulas assert

$$\chi_T(F) = ( ) + \alpha$$

$$\beta = ( ) + 2\alpha$$

where the terms in parentheses depend only on  $V$  (and not on the smoothing). Thus  $\alpha$ , which is a priori an analytic invariant, depends only on the topology of the smoothing, hence is the same on a smoothing component of the moduli space. In particular, according to the formulas, if one smoothing is  $\omega^*$ -constant, so are all smoothings on the same irreducible component.

(3.14.3) For a smoothing of negative weight when there is a  $\mathbb{C}^*$ -action, the above formulas are valid (3.3.d). Further,  $b_1(F) = 0$  (2.2), so  $\mu_0, \mu_+$  are both even. (This was discovered by Steenbrink for hypersurfaces [32].) For,  $\mu_0 = b_1(E)$  is even, as follows from the star-shaped configuration of the graph  $E$ .

(3.14.4) If  $V$  is Cohen–Macaulay of codimension 2, then the formulas are valid (3.3.e and 3.4) with  $\alpha = 0$ . For, Buchsbaum and Eisenbud have proved [4] that if  $R = P/I$ ,  $P$  a regular ring, then  $I/I^2$  is torsion-free; since  $\text{Ext}_R^1(\omega, R) = \text{torsion submodule of } I/I^2$ , it is 0. Thus,  $\omega^*$ -constant smoothing is automatic.

(3.15) If  $R$  is Gorenstein, then (a), (b), and (d) reduce to known formulas of Durfee [9] and Laufer [20]. First,  $\alpha = 0$ . Second,  $K_X = \omega_X = \mathcal{O}(-Z)$ , for some effective cycle  $Z$ . From the exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow \omega_X^* = \mathcal{O}_X(Z) \rightarrow \mathcal{O}_Z(Z) \rightarrow 0,$$

the vanishing  $H^0(\mathcal{O}_Z(Z)) = 0$ , and  $Z = -K$ , Riemann–Roch gives  $h^1(\mathcal{O}_Z(Z)) = -Z \cdot Z$ ; thus,

$$h^1(\omega_X^*) = h^1(\mathcal{O}_X) + Z \cdot Z.$$

Plugging into (3.13) gives

$$(3.15.1) \quad \chi_T(F) = 12h^1(\mathcal{O}_X) + \chi_T(E) + Z \cdot Z$$

$$(3.15.2) \quad \sigma(F) = -8h^1(\mathcal{O}_X) - Z \cdot Z - r$$

$$(3.15.3) \quad \beta = h^1(\theta_X) + 10h^1(\mathcal{O}_X) + 2Z \cdot Z.$$

Note all formulas here are independent of the smoothing.

**COROLLARY (3.16).** *The formulas (3.15.1)–(3.15.3) are valid for a complete intersection, or a Gorenstein  $V$  of codimension 3. In these cases,  $\beta$  = dimension of moduli space of  $V$ .*

*Proof.* Globalizability is given in (3.3). The interpretation of  $\beta$  is given in (4.4) below.

(3.17). If  $V$  is rational ( $h^1(\mathcal{O}_X) = 0$ ), we have  $b_1(E) = 0$ , so  $b_1(F) = 0$ , and we can write  $\chi_T(F) = 1 + \mu$ . In [35], we introduced the invariant

$$q(R) = l(\text{Ext}_R^1(\omega, R)),$$

called  $l(R)$  there, and proved

$$(3.17.1) \quad q(R) = \dim H^1(\omega_X^*) \geq \text{mult } R - 3.$$

For smoothings on the Artin component  $A$  [34], one has

$$(3.17.2) \quad \begin{aligned} \mu &= \mu_- = r \\ \sigma &= -r \\ \dim A &= h^1(\theta_X). \end{aligned}$$

**COROLLARY 3.18.** *For a smoothing of a rational singularity which globalizes (3.1.1), we have*

(a)  $\mu = r - q(R) + \alpha$ , where  $0 \leq \alpha \leq q(R)$ , and  $\alpha = 0$  iff the smoothing is  $\omega^*$ -constant.

(b)  $\mu_0 = \mu_+ = 0$ , and  $\sigma = -\mu$

(c)  $\beta = h^1(\theta_X) - 2(q(R) - \alpha)$

(3.19) Note (3.13) implies  $\alpha = q(R)$  on  $A$ ; we prove this independently of the globalization hypothesis.

**THEOREM 3.20.** *For an Artin component smoothing,  $\alpha = q(R)$ ; in particular, the smoothing is not  $\omega^*$ -constant once  $\text{mult } R \geq 4$ .*

*Proof.* Let  $\mathcal{V} \rightarrow T$  be the smoothing in question. Then without base change there is a simultaneous rational double point (RDP) resolution  $\mathcal{X} \xrightarrow{f} \mathcal{V} \rightarrow T$  [22], with  $X_t$  smooth ( $t \neq 0$ ), and  $X_0$  possessing only RDP's. If  $P \in \mathcal{V}$  is the unique singular point then  $f^{-1}(P) = \mathcal{C} \subset \mathcal{X}$  is a curve; let  $U$  denote the complement,  $i: U \rightarrow \mathcal{V}$ . Lipman proves ([22], Lemma 3) that

$$\mathcal{X} = \text{Proj} \left( \bigoplus_{n \geq 0} f_*(\omega_{\mathcal{X}/T}^{\otimes n}) \right)$$

and there is an inclusion

$$(*) \quad \theta_n: f_*(\omega_{\mathcal{X}/T}^{\otimes n}) \rightarrow i_*(\omega_{U/T}^{\otimes n}).$$

We claim  $(*)$  is an isomorphism for all  $n \geq 0$ ; but this follows because the  $\mathcal{X} - U$  depth of  $\omega_{\mathcal{X}/T}^{\otimes n}$  is  $\geq 2$  (this complement is a curve, and  $\mathcal{X}$  is normal). So,

$$\omega^n = f_*(\omega_{\mathcal{X}/T}^{\otimes n})$$

has depth  $\geq 2$  for all  $n \geq 0$ . The natural surjection

$$\omega_{\mathcal{V}/T}^{\otimes n} \rightarrow \omega^n$$

therefore gives

$$H_P^1(\omega_{\mathcal{V}/T}^{\otimes n}) = 0, \quad n \geq 0.$$

By local duality,

$$\text{Ext}_{\mathcal{V}}^2(\omega_{\mathcal{V}/T}^{\otimes n}, \omega_{\mathcal{V}}) = 0, \quad n \geq 0.$$

But  $\omega_{\mathcal{V}}$  is (non-canonically) isomorphic to  $\omega_{\mathcal{V}/T}$ , as they have depth  $\geq 2$  and are isomorphic on  $U$  (since  $\omega_T$  is free). Thus,

$$\text{Ext}_{\mathcal{V}}^2(\omega_{\mathcal{V}/T}^{\otimes n} \otimes \omega_{\mathcal{V}}, \omega_{\mathcal{V}}) = 0, \quad n \geq 0.$$

An exercise shows these are isomorphic to

$$\text{Ext}_{\mathcal{V}}^2(\omega_{\mathcal{V}/T}^{\otimes n}, \mathcal{O}_{\mathcal{V}}) = 0, \quad n \geq 0.$$

Now, using  $n = 1$  in Theorem A.1(a) gives

$$\alpha = l(\text{Ext}_{\mathcal{V}}^1(\omega_{\mathcal{V}}, \mathcal{O}_{\mathcal{V}})) = q(R),$$

as desired. The last comment follows from (3.17.1).

*Example (3.21).* If  $R$  is the cone over a rational quartic curve  $\subset \mathbb{P}^4$ , one has  $r = 1$ ,  $q(R) = 1$ ; Pinkham has shown [26] the moduli space has 2 components.  $A$  has dimension 3, with  $\mu = \alpha = 1$ ; another component has dimension 1, with  $\mu = \alpha = 0$ .

(3.22) The analysis (3.7)–(3.13) could have been extended to consider  $\Omega_{V/T}^1$ . If  $R$  is a complete intersection, or Gorenstein of codimension 3, one can prove

$$\beta = 10h^1(\mathcal{O}_X) + h^1(\Omega_X^1) + Z \cdot Z + u,$$

where  $u = l(\text{Coker } H^0(X, \Omega_X^1) \rightarrow H^0(X - E, \Omega_X^1))$  is the number of conditions on a 1-form on  $X - E$  that it extend holomorphically over  $E$ . Recall  $\beta = \dim T_R^1$  in these cases.

(3.2.3) The same method of proof as above yields formulas for  $\mu$  and  $\sigma$  in the higher-dimensional case, especially for hypersurfaces ([2, 36]). One compares  $\chi(\Omega^p)$  on  $Y$ ,  $Y$ , and  $\bar{X}$ , using the de Rham lemma to give flatness of (most)  $\Omega_{Y/T}^p$  over  $T$ ; then use that in dimension  $n$ ,

$$\chi_T = \sum_{p=0}^n (-1)^p \chi(\Omega^p).$$

#### §4. THE DIMENSION OF SMOOTHING COMPONENTS

(4.1) Let  $\mathcal{V}^* \rightarrow S^*$  be the (mini-)versal deformation of the space  $V$  (with isolated singularity of arbitrary dimension). If  $T_V^1$ ,  $T_V^2$  are respectively the tangent and obstruction spaces to the deformations, it is known (e.g. [39], Theorem 5.11) that  $S^*$  is the fibre over 0 of a non-canonical map-germ of  $T_V^1 \rightarrow T_V^2$ ; it follows that any irreducible component  $S_i$  of  $S^*$  has dimension between  $l(T_V^1) - l(T_V^2)$  and  $l(T_V^1)$ . A component  $S_i$  of  $S^*$  is called a smoothing component of  $V$  if the general fibre of  $\mathcal{V}^* \rightarrow S^*$  over  $S_i$  is smooth; by local versality,  $S_i$  is generically smooth. Any one-parameter smoothing  $\mathcal{V} \rightarrow T$  “lies on” a unique smoothing component. As in (3.8), there is an inclusion

$$\theta_{V/T} \otimes \mathcal{O}_V \subset \theta_V,$$

whose cokernel has finite length  $\beta$ .

CONJECTURE 4.2.  $\beta = \text{dimension of smoothing component on which } \mathcal{V} \rightarrow T \text{ lies.}$

THEOREM 4.3. (a)  $\beta > 0$ ; (b)  $\beta \leq l(T_V^1)$ ; if  $V$  is normal, then  $l(T_V^1) - l(T_V^2) \leq \beta$ ; (c) Conjecture 4.2 is valid if  $V$  can be globalized to a projective  $Y$ , non-singular away from  $V$ , with  $H^2(\theta_Y) = H^2(\mathcal{O}_Y) = 0$ .

*Proof.* The smoothing  $\pi: \mathcal{V} \rightarrow T$  gives

$$\pi^* \Omega_T^1 \rightarrow \Omega_{\mathcal{V}}^1 \rightarrow \Omega_{V/T}^1 \rightarrow 0,$$

with the first map having torsion kernel; dualizing gives

$$\theta_V \rightarrow \pi^* \theta_T \rightarrow \text{Ext}_{\mathcal{V}}^1(\Omega_{V/T}^1, \mathcal{O}_V).$$

Now,  $\beta = 0$  implies the last term is zero, by Theorem A. 1(b) (of the Appendix) applied to  $\Omega_{Y/T}^1$  and  $n = 0$ . Therefore, the derivation  $(d/dt)$  of  $\theta_T$  lifts to a derivation  $D$  of  $\theta_Y$ . If  $A$  denotes the complete local ring of  $\mathcal{V}$  at the singular point, we have that  $D: A \rightarrow A$  is a derivation, inducing  $(d/dt)$  on the subring  $\mathbb{C}[[t]]$ . In particular,  $D(m_A) \not\subset m_A$ ; thus, by a well-known exponentiation trick of Seidenberg–Zariski (e.g. [33], p. 586), one can “integrate”  $D$ , obtaining

$$A \simeq A_1[[t]]$$

for some subring  $A_1$ . This is impossible for a smoothing, so  $\beta > 0$ , and (a) is proved.

For (b), apply Corollary A.2 to  $\Omega_{Y/T}^1$ , since  $T_V^1 = \text{Ext}_V^1(\Omega_V^1, \mathcal{O}_V)$ , and (if  $V$  is normal)  $T_V^2 = \text{Ext}_V^2(\Omega_V^1, \mathcal{O}_V)$  [41].

For (c), choose an embedding  $Y \subset \mathbb{P}^N = \mathbb{P}$  with  $H^i(\mathcal{O}_Y(1)) = 0$ ,  $i = 1, 2$ . Since  $H^2(\mathcal{O}_Y) = 0$ ,  $H^1(\theta_{\mathbb{P}} \otimes \mathcal{O}_Y) = 0$ , so all deformations of  $Y$  occur in  $\mathbb{P}$ . Since  $H^2(\theta_Y) = 0$ , deformations of  $Y$  map smoothly (by restriction) to deformations of  $V$ . Thus, if  $H^*$  is the Hilbert scheme of  $\mathbb{P}$  at  $Y$ , then  $H^* \rightarrow S^*$  is smooth. (Note (3.1.1) is therefore satisfied). For  $h \in H^*$ , one has the exact sequence

$$(4.3.1) \quad 0 \rightarrow \theta_{Y_h} \rightarrow \theta_{\mathbb{P}} \otimes \mathcal{O}_{Y_h} \rightarrow N_{Y_h} \rightarrow T_{Y_h}^1 \rightarrow 0;$$

define also

$$N'_{Y_h} = \text{Ker}(N_{Y_h} \rightarrow T_{Y_h}^1).$$

Note  $H^1(N'_Y) = 0$ . The map  $H^* \rightarrow S^*$  has fibre dimension

$$h^0(N'_Y).$$

A smoothing component  $S_1 \subset S^*$  gives rise to an  $H_1 \subset H^*$ ; and

$$\dim H_1 = h^0(N_{Y_h})(h \in H_1 \text{ generic}),$$

as  $H_1$  is generically smooth. So,

$$(4.3.2) \quad \dim S_1 = h^0(N_{Y_h}) - h^0(N'_Y).$$

Choose a one-parameter smoothing  $\mathcal{Y} \rightarrow T$  in  $H_1$  (or mapping finitely onto  $H_1$ ); let  $Y_t \subset \mathbb{P}$  be the generic fibre. Define a sheaf  $\mathcal{N}$  on  $\mathcal{Y}$  by

$$(4.3.3) \quad 0 \rightarrow \theta_{\mathcal{Y}/T} \rightarrow \theta_{\mathbb{P} \times T/T} \otimes \mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{N} \rightarrow 0.$$

$\mathcal{N}$  is flat over  $T$ , being contained in the torsion-free (dual) normal sheaf  $N_{\mathcal{Y}/\mathbb{P}}$ ;  $\mathcal{N}$  induces  $N_{Y_t}$  for  $t \neq 0$ . Tensoring (4.3.3) with  $\mathcal{O}_Y$  gives

$$\begin{array}{ccccccc} 0 \rightarrow \theta_{\mathcal{Y}/T} \otimes \mathcal{O}_Y & \rightarrow & \theta_{\mathbb{P}} \otimes \mathcal{O}_Y & \rightarrow & \mathcal{N} \otimes \mathcal{O}_Y & \rightarrow & 0 \\ \downarrow & & \parallel & & \downarrow & & \\ 0 \rightarrow \theta_Y & \rightarrow & \theta_{\mathbb{P}} \otimes \mathcal{O}_Y & \rightarrow & N'_Y & \rightarrow & 0. \end{array}$$



Using the definition of  $\beta$  and  $H^1(N'_Y) = 0$ , we deduce

$$\begin{aligned} h^1(\mathcal{N} \otimes \mathcal{O}_Y) &= 0 \\ \beta &= h^0(\mathcal{N} \otimes \mathcal{O}_Y) - h^0(N'_Y). \end{aligned}$$

By semi-continuity applied to  $\mathcal{N}$ ,  $h^1(N_{Y_t}) = 0$ ,  $h^0(N_{Y_t}) = h^0(\mathcal{N} \otimes \mathcal{O}_Y)$ . Combining with (4.3.2) gives the result.

**COROLLARY 4.4.** *If  $T_V^2 = 0$  and  $V$  is normal, then Conjecture 4.2 is true;  $\beta$  is the dimension of the (smooth) space  $S^*$ .*

*Proof:* (4.3.b).

**COROLLARY 4.5** (Deligne [7], Exp. X). *The dimension of a smoothing component of a curve singularity  $R$ , with normalization  $\tilde{R}$ , is equal to*

$$\beta = 3\delta - l(\theta_{\tilde{R}}/\theta_R).$$

*Proof.* Since a curve can be globalized in the sense of (4.3.c), by the theorem we need only compute  $\beta$  for a one-parameter smoothing. Using the notation of (3.6) and the technique of (3.7)–(3.12), we write

$$\begin{aligned} \chi(\theta_{Y_t}) &= -3\chi(\mathcal{O}_{Y_t}) \\ \chi(\theta_{\tilde{Y}}) &= -3\chi(\mathcal{O}_{\tilde{Y}}). \end{aligned}$$

Note  $\theta_Y \subset \theta_{\tilde{Y}}$ , since derivations extend to total quotient rings in characteristic zero; naturally,

$$\chi(\theta_{\tilde{Y}}) - \chi(\theta_Y) = l(\theta_{\tilde{R}}/\theta_R).$$

From the semicontinuity theorem,

$$\chi(\theta_Y) - \chi(\theta_{Y_t}) = \beta.$$

Putting everything together gives Deligne's result.

**COROLLARY 4.6.** *Let  $V$  be a normal surface singularity for which Theorem 3.13 and Conjecture 4.2 are valid (e.g. a complete intersection, or Cohen–Macaulay of codimension 2, or Gorenstein of codimension 3, or a quotient singularity  $\mathbb{C}^2/G$ ). If  $S \subset S^*$  is a smoothing component, with Milnor fibre  $F$ , then*

$$\begin{aligned} (4.6.1) \quad \dim S &= h^1(\theta_X) - 14h^1(\mathcal{O}_X) + 2(\chi_T(F) - \chi_T(E)) \\ &= h^1(\theta_X) + 12h^1(\mathcal{O}_X) - 2(h^1(\omega_X^*) - \alpha). \end{aligned}$$

*If  $V$  is Gorenstein, then*

$$(4.6.2) \quad \dim S = h^1(\theta_X) + 10h^1(\mathcal{O}_X) + 2Z \cdot Z.$$

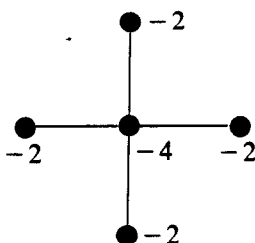
*Proof.* For the formulas, use Theorem 3.13. For the applicability of the examples, use Theorem 3.3 and 4.3, plus (for the quotients) 6.5.

*Remarks (4.7.1).* The formula (4.6.2) for a hypersurface was found independently by S. S.-T. Yau.

(4.7.2) The formula (4.6.1) implies that for two smoothing components,  $\dim S_i - \dim S_j$  is even.

(4.7.3) For a rational singularity, (3.18) and (4.6) imply the Artin component has maximal dimension among the  $S_i$ ; we have proved this for the quotients and, more generally, for singularities with  $\mathbb{C}^*$ -action all of whose deformations have weight  $\leq 0$ . This was conjectured by Riemenschneider in some cases. In fact, using Riemenschneider's calculation of  $l(T_R^1)$  for cyclic and dihedral quotients ([28, 29]), one can prove that in these cases the Artin component has larger dimension (by at least 2) than any other component.

(4.7.4) A rational quadruple point with dual graph



(see [34], 3.3) has 3 components of  $S^*$ , of dimensions 8, 6, and 4.

(4.8) It is obvious that for a hypersurface  $V$ ,  $\mu \geq \tau = \dim T^1$ , with equality if  $V$  has a  $\mathbb{C}^*$ -action. The generalization of this second fact to complete intersections is not at all trivial, and was proved by Greuel [11]. We offer the

**CONJECTURE 4.9.** *Let  $V$  be a Gorenstein surface singularity with  $\mathbb{C}^*$ -action,  $S \subset S^*$  a smoothing component, and  $F$  the corresponding Milnor fibre. Then*

$$\dim S = \chi_T(F) - 1 (= \mu, \text{ if } b_1(F) = 0).$$

**THEOREM 4.10.** *Assuming Conjecture 4.2 and the validity of Theorem 3.13 (e.g.  $V$  is a complete intersection, or Gorenstein of codimension 3), the Conjecture 4.9 is true.*

*Proof.* By (3.15.1)–(3.15.3) and (4.6.2), we must prove

$$(4.10.1) \quad h^1(\theta_X) = 2h^1(\mathcal{O}_X) + \chi_T(E) - Z \cdot Z - 1,$$

where  $\omega_X = \mathcal{O}_X(-Z)$ . Excluding the case of rational double points (which is easily verified), we may assume the graph is star-shaped, with a central curve  $E_1$  and (outer) rational curves  $E_i$ ,  $i \geq 2$ . (We work with a non-minimal resolution if  $E_1 \cdot E_1 = -1$ ,  $E_1$  rational, but this does not matter.) Recall the rank 2 vector bundle on  $X$  defined by

$$0 \rightarrow S \rightarrow \theta_X \rightarrow \bigoplus N_{E_i} \rightarrow 0.$$

Also, the  $C^*$ -action gives rise to a derivation  $D$  in  $\theta_V$ , hence in  $\theta_X$  ( $D = \sum a_i x_i (\partial/\partial x_i)$ ,  $a_i$ 's = weights). In fact, a local check shows  $D \in H^0(S)$ , and there is a exact sequence

$$(4.10.2) \quad 0 \rightarrow \mathcal{O}_X \xrightarrow{D} S \xrightarrow{\wedge^2 D} \Lambda^2 S \rightarrow 0$$

of vector bundles on  $X$ . One also shows easily

$$\Lambda^2 S \simeq \mathcal{O}_X(Z - E),$$

where  $E$  is the (reduced) exceptional fibre. Note  $E \leq Z$ , since we exclude rational double points.

A subtle but important fact is that the cohomology map from (4.10.2)

$$(4.10.3) \quad H^0(\Lambda^2 S) \rightarrow H^1(\mathcal{O}_X)$$

has image of dimension exactly 1. Assuming this for a moment, we deduce

$$h^1(S) = h^1(\mathcal{O}_X) + h^1(\mathcal{O}_X(Z - E)) - 1.$$

But  $h^0(\mathcal{O}_{Z-E}(Z - E)) = 0$  (as  $Z - E \geq 0$ ), so

$$h^1(S) = 2h^1(\mathcal{O}_X) + h^1(\mathcal{O}_{Z-E}(Z - E)) - 1$$

Back to (4.10.1), it now suffices to prove

$$h^1(\mathcal{O}_{Z-E}(Z - E)) + \sum h^1(N_{E_i}) = \chi_T(E) - Z \cdot Z.$$

But this is a straightforward computation, which we leave to the reader.

Returning to (4.10.3), compare the map with the one arising from tensoring (4.10.2) with  $\mathcal{O}_Z$ :

$$(4.10.4) \quad \begin{array}{ccc} H^0(\mathcal{O}_X(Z - E)) & \rightarrow & H^1(\mathcal{O}_X) \\ \downarrow & & \downarrow \\ H^0(\mathcal{O}_Z(Z - E)) & \rightarrow & H^1(\mathcal{O}_Z). \end{array}$$

The right vertical map is an isomorphism because  $H^1(\mathcal{O}_X(-Z)) = H^1(\omega_X) = 0$  (Grauert–Riemenschneider). But  $h^0(\mathcal{O}_Z(Z - E)) = 1$ , as follows from

$$0 \rightarrow \mathcal{O}_E \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z-E} \rightarrow 0.$$

Thus, the map (4.10.3) has one-dimensional image, or is the 0-map. But the latter possibility gives

$$\begin{array}{ccccccc} 0 \rightarrow H^0(\mathcal{O}_X) & \rightarrow & H^0(S) & \rightarrow & H^0(\mathcal{O}_X(Z - E)) & \rightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \\ R & & \theta_R & & R & & \end{array}$$

whence  $\theta_R$  is free; but this contradicts the Zariski–Lipman conjecture in the graded case [15].

*Remark (4.11).* A more careful analysis of (4.10.4) actually proves the Zariski–Lipman conjecture in dimension 2 (graded case). Note one has an exact sequence (dimension 2, Gorenstein, graded)

$$\mathcal{O} \rightarrow R \xrightarrow{D} \theta_R \rightarrow R \rightarrow R/m = \mathbb{C} \rightarrow 0,$$

whence  $\theta_R$  requires at least  $e$  generators as an  $R$ -module.

#### §5. EXAMPLES AND CONJECTURES

(5.1) In [20], Laufer conjectured that the Milnor fibre of a smoothing of a Gorenstein surface satisfies

$$(5.1.1) \quad \chi_T(F) > 0;$$

rather, he asserted that the expression (from 3.15.1)

$$(5.1.2) \quad 12h^1(\mathcal{O}_X) + \chi_T(E) + Z \cdot Z > 0.$$

Note (5.1.1) would follow easily from  $b_1(F) = 0$ . Further, globalization as in (3.1.1) yields (via (3.13.d))

$$\mu_+ + \mu_0 = 2h^1(\mathcal{O}_X).$$

The “correct” form of (5.1.1) should therefore be

$$(5.1.3) \quad \chi_T(F) \geq 2h^1(\mathcal{O}_X) + 1,$$

with equality iff  $\mu_- = 0$ . We summarize these, using §3, in

**CONJECTURE 5.2.** *If  $F$  is the Milnor fibre of a smoothing of a normal surface singularity  $V$ , then*

- (a)  $b_1(F) = 0$
- (b)  $\mu_+ + \mu_0 = 2h^1(\mathcal{O}_X)$
- (c)  $\mu_- = 11h^1(\mathcal{O}_X) + \chi_T(E) - h^1(\omega_X^*) + \alpha - 1.$

*If further  $V$  is Gorenstein, then*

$$(d) \quad \mu_- = 10h^1(\mathcal{O}_X) + \chi_T(E) + Z \cdot Z - 1.$$

(5.3) Since  $\mu_- \geq 0$ , (c) and (d) yield inequalities which must be satisfied for  $V$  to be smoothable. Of course (c) is hard to use, as  $\alpha \leq q(R)$ ; but there is an obvious restriction to have an  $\omega^*$ -constant smoothing ( $\alpha = 0$  in that case).

**PROPOSITION 5.4.** *Conjecture 5.2 is satisfied if  $V$  has a  $\mathbb{C}^*$ -action, and the smoothing has negative weight.*

*Proof.* Use (2.2), (3.3.d), and Theorem 3.13.

(5.5). Therefore, a Gorenstein singularity with  $C^*$ -action cannot be smoothed negatively if

$$(5.5.1) \quad 10h^1(\mathcal{O}_X) + \chi_T(E) + Z \cdot Z - 1 < 0.$$

This gives a theorem of Pinkham ([27], Corollary 6.14) when applied to a certain class of singularities (e.g. Dolgachev singularities); his proof used that, after globalizing, a nearby smooth fibre is a minimal  $K$ -3 surface.

**THEOREM 5.6.** *Let  $V$  be a cusp for the Hilbert modular group, of multiplicity  $m$ , and with  $r$  exceptional curves in the minimal resolution. If  $V$  is smoothable, then*

$$m \leq r + 9.$$

*Proof.* For a smoothing,  $\beta > 0$  (4.3.a); since cusp deformations can be globalized (3.3.c), this inequality may be written (3.15.3)

$$h^1(\theta_X) + 10h^1(\mathcal{O}_X) + 2Z \cdot Z > 0.$$

Now,  $h^1(\theta_X) = \sum h^1(N_{E_i})$  since  $V$  is taut ([19]); if  $E_i \cdot E_i = -d_i$ ,

$$h^1(\theta_X) = \sum (d_i - 1) = m + r.$$

A cusp is minimally elliptic [40], so  $h^1(\mathcal{O}_X) = 1$  and  $m = \max(-Z \cdot Z, 2)$ . If  $-Z \cdot Z = 1$ , then  $m = 2$ , and  $2 \leq r + 9$  always; if  $-Z \cdot Z = m$ , the above inequality becomes

$$r + 10 - m > 0.$$

(5.7) By examples (5.9.4) below, the inequality is sharp. It is proved in [35] that a cusp is smoothable if  $r > m^2 - m$ . The cusps with  $m \geq r + 10$  are the only non-smoothable singularities we know of which do not have  $C^*$ -action.

(5.8) In [35], 2.10, we gave a construction for some  $\omega^*$ -constant smoothings. Let  $G \subset GL(3, \mathbb{C})$  be a finite cyclic (diagonal) group, acting freely on  $\mathbb{C}^3 - \{0\}$ ,  $f \in \mathbb{C}[x, y, z]$  a  $G$ -invariant polynomial, and  $H =$  the hypersurface  $\{f = 0\}$ . Then  $f: \mathbb{C}^3/G \rightarrow \mathbb{C}$  defines an  $\omega^*$ -constant smoothing of the fibre, which is the hypersurface quotient  $V = H/G$ . If  $F_H$  and  $F = F_V$  are the corresponding Milnor fibres, then  $G$  acts freely on  $F_H$ , and

$$F_H/G = F;$$

so, if  $m = |G|$ , then

$$m\chi_T(F) = \chi_T(F_H).$$

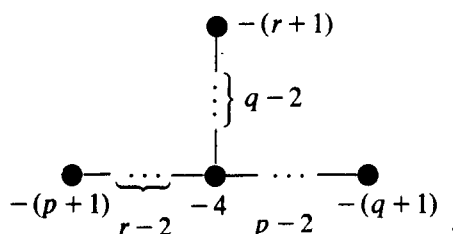
Further, since  $\pi_1(F_H) = 0$  ([23]), we have  $\pi_1(F) = G$ , whence

$$b_1(F) = 0.$$

If  $f$  is weighted homogeneous, then  $\mathcal{R} = \mathbb{C}[x, y, z, t]/t - f$  is non-negatively graded, hence so is  $\mathcal{R}^G$ ; the induced deformation of  $V$  is therefore of negative weight, whence Theorem 3.13 holds. In the following examples (of  $F$  and  $G$ ), the structure of  $V$  is found by examining the action of  $G$  on a resolution of  $H$ .

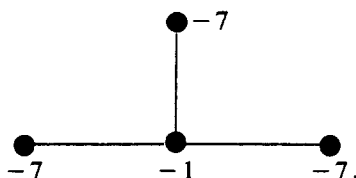
Examples (5.9.1) ([35], 2.10) Let  $0 < q < n$ ,  $(q, n) = 1$ , and  $f = xz - y^n$ . The cyclic group of order  $n$  acts via  $[\omega, \omega^q, \omega^{n-1}]$ , where  $\omega$  is a primitive  $n^{\text{th}}$  root of 1. Then  $V$  is the  $(n^2, nq - 1)$  quotient singularity. As  $\chi_T(F_H) = n$ ,  $\chi_T(F) = 1$ ; thus,  $\mu = 0$ , and  $F$  is a rational homology ball. ( $\partial F$  is a lens space which is well-known to bound such a manifold.) This 1-parameter smoothing is an irreducible component of the deformation space of  $V$ , since §3 and §4 yield  $\beta = 1$ .

(5.9.2). Let  $f = xy^p + yz^q + zx^r$ ; then  $f$  is weighted homogeneous, with  $\mu = pqr$ . Let  $\omega$  be a primitive  $N^{\text{th}}$  root of 1, where  $N = pqr + 1$ , and let  $G$  act via  $T = [\omega, \omega^{qr}, \omega^{N-r}]$ . As  $\chi_T(F_H) = N$ , the quotient  $V$  again has  $\chi_T(V) = 1$ , so  $\mu = 0$ . In particular,  $F_H/G$  is a rational homology ball bounding the link of  $V$  (actually, this case includes 5.9.1).  $V$  is a rational singularity with graph (if  $p, q, r \geq 2$ , and where unmarked dots are  $-2$  curves):



Again, the smoothing is a full component of  $S^*$ . This triply-infinite family of compact 3-manifolds bounding rational homology balls is part of a triply-infinite family discovered recently (and independently) by Neumann[25] using techniques from differential topology.

(5.9.3). In the last example, let  $p = q = r = 3$  (the cone over the Klein quartic), and  $S = T^4$ . Dividing by  $\{S\}$  gives the Dolgachev singularity  $V = Tr(6, 6, 6)$ :



Note  $\mu = 3$ ; as  $\mu_0 + \mu_+ = 2$ ,  $\sigma = \mu_+ - \mu_- = 1$ , we deduce  $\mu_0 = 0$ ,  $\mu_+ = 2$ ,  $\mu_- = 1$ . That the signature is positive contradicts a conjecture of Durfee ([9], 5.2).

(5.9.4) Let  $\omega$  be a primitive  $n^{\text{th}}$  root of 1 ( $n \geq 3$ ). Consider the cyclic group generated by  $T = [\omega, \omega^a, \omega^b]$ , where  $(a, n) = (b, n) = 1$  and  $a + b + 1 = n$  (so,  $n$  is odd). Let  $f = xyz + x^n + y^n + z^n$ . Globalization applies to the smoothing  $f = t$  and the quotient, since  $f = t$  is a deformation in the negative part of the versal deformation of  $x^n + y^n + z^n = 0$ . The quotient  $V$  is a cusp with  $\mu = 2$  and (5.6)  $m = r + 9$ . In fact,  $\mu_+ = \mu_0 = 1$ ,  $\mu_- = 0$ , so the intersection pairing is positive semi-definite.

(5.10) Assuming strong globalization (4.3.c), the irreducible components of the moduli space of a rational singularity have dimension

$$h^1(\theta_X) - 2i,$$

for some  $0 \leq i \leq q(R)$ . The Artin component has  $i = 0$ ; an  $\omega^*$ -constant smoothing

component, if it exists, has  $i = q(R)$ . A general method for constructing deformations of  $R$  is as follows: Let  $X' \rightarrow V$  be a partial resolution [35];  $X'$  is allowed to have rational singularities. As  $h^1(\mathcal{O}_{X'}) = 0$ , the  $\omega^*$ -constant deformations of  $X'$  blow down to deformations of  $V$ . Perhaps all deformations of  $R$  arise this way.

#### §6. GLOBALIZABILITY

(6.1) In this section we prove globalizability results in the following cases: determinantal (6.2); Pfaffian (6.3);  $H^2(\theta) = 0$  globalization (6.4); cusp for the Hilbert modular group (6.5); and quotient singularity (6.6).

**THEOREM 6.2** (see [18]). *Let  $V$  be an isolated determinantal singularity, defined by the  $t \times t$  minors of an  $r \times s$  matrix  $(f_{ij})$  of formal power series, with  $2 \leq t \leq r \leq s$ . If  $\dim V < s + r - 2t + 3$ , then  $V$  is smoothable determinantly, and such a smoothing may be globalized in the sense of (3.1.1).*

*Proof.* Let  $e = \text{emb dim } V$ ; then it is well-known that

$$e - \dim V = (r - t + 1)(s - t + 1).$$

As the singularity is isolated (hence algebraic), one may assume there is an integer  $N$  so that

- (i) the  $f_{ij}$  are polynomials of degree  $< N$
- (ii) addition of monomials of degree  $\geq N$  to the  $f_{ij}$  (which automatically gives a flat deformation of the singularity) leave the singularity unchanged.

Thus,  $V$  is now an affine variety, giving the original singularity at 0, but possibly with other singularities.

Let  $M$  be the number of monomials of degree  $N$  in  $e$  variables,  $T = \mathbb{C}^{rsM}$ , and  $\mathcal{V} \rightarrow T$  the morphism defined by the  $t \times t$  minors of  $(F_{ij}) = (f_{ij} + t_{ij}^{(1)}x_1^N + \dots + t_{ij}^{(M)}x_e^N)$ . By construction, there is a section  $\sigma: T \rightarrow \mathcal{V}$  (given by  $x_1 = \dots = x_e = 0$ ) landing in the singular locus  $\text{Sing}(\mathcal{V})$ , along which  $\mathcal{V} \rightarrow T$  is formally trivial. Denote by  $D_k \subset T \times \mathbb{C}^e$  the subscheme defined by the  $k \times k$  minors of  $(F_{ij})$ . So,  $D_t = \mathcal{V}$ ,  $D_k \subset D_{k+1}$ , and we may assume  $\sigma(T) \subset D_1$  (i.e. the  $f_{ij}$  all vanish at 0). We assert

$$(6.2.1) \quad \text{Sing}(D_{k+1}) \subset D_k, \quad i \leq k \leq t-1$$

$$(6.2.2) \quad D_1 - \sigma(T) \text{ is non-singular, of dimension } e + rsM - rs$$

$$(6.2.3) \quad \dim(D_{k+1} - \sigma(T)) \leq e + rsM - (r-k)(s-k).$$

For (6.2.1), we pick a point  $z$  of  $D_{k+1} - D_k$ , and show  $D_{k+1}$  is non-singular there. We may as well assume the upper left hand  $k \times k$  minor  $G = G(X, t_{ij}^{(e)})$  is invertible at  $z$ ; note  $1 \leq i, j \leq k$ . Let  $g_{\alpha\beta}$  be the  $(k+1) \times (k+1)$  minor obtained by adding row  $\alpha$  and column  $\beta$  to  $G$  (note  $k+1 \leq \alpha \leq r$ ;  $k+1 \leq \beta \leq s$ ). Since  $G \neq 0$ , some  $x_j \neq 0$ , say  $x_1$ . Therefore,

$$g_{\alpha\beta} = t_{\alpha\beta}^{(1)}x_1^N G + (\text{terms involving no } t_{ij}^{(1)}, i \text{ and } j \geq k+1).$$

Applying the Jacobian criterion to the  $\{g_{\alpha\beta}\}$ , differentiated with respect to the  $\{t_{\alpha\beta}\}$ ,

$k + 1 \leq \alpha \leq r$ ,  $k + 1 \leq \beta \leq s$ , gives that the dimension of the Zariski tangent space of  $D_{k+1}$  at  $z$  is

$$\dim T_{D_{k+1},z} \leq e + rsM - (r - k)(s - k).$$

However, since  $D_{k+1}$  is defined by determinants, it is well known that

$$\dim D_{k+1,z} \geq e + rsM - (r - k)(s - k).$$

Therefore,  $D_{k+1}$  is non-singular at  $z$ , of the given dimension. This gives (6.2.1); and (6.2.2) is similar, but easier. Therefore, every component of  $D_{k+1}$  has dimension less than or equal to the asserted value, whence (6.2.3).

In particular, with  $k = t - 2$

$$(6.2.4) \quad \dim(S(\mathcal{V}) - \sigma(T)) \leq e + \dim T - (r - t + 2)(s - t + 2).$$

But

$$e - (r - t + 2)(s - t + 2) = \dim V - (s + r + 3 - 2t) < 0,$$

by assumption. Therefore

$$(6.2.5) \quad \dim(S(\mathcal{V}) - \sigma(T)) < \dim T.$$

It follows that over a non-empty open set  $T'$  of  $T$ ,  $\mathcal{V} - \sigma(T)$  is non-singular; we can further assume  $\mathcal{V}' \rightarrow T'$  is flat, of constant fibre dimension. By Bertini's theorem (see, e.g. [33], 2.8, for a simple proof of a strong version), the generic fibre  $V_t$  is an affine determinantal variety, non-singular off the one point at which it is isomorphic to  $V$ .

Similarly, homogenizing the  $f_{ij}$  (or  $F_{ij}$ ) by addition of a new variable  $x_0$ , and dehomogenizing by setting some  $x_i = 1$ , one obtains  $Y_t \subset \mathbb{P}^e$ , a globalization of  $V$  (or  $V_t$ ) which is non-singular off  $\sigma(t)$ , and which is projectively determinantal.

We may suppose now  $V = V_t$  has isolated singularity and globalizes to a  $Y$  as above. Form  $\mathcal{V}$  as above with  $M = 1$  (i.e. consider  $(F_{ij} + t_{ij})$ ). The same argument given before shows this yields a smoothing of  $V$  for generic  $t$ , and it certainly globalizes by construction. This completes the proof.

**THEOREM 6.3** (see [3]). *Let  $V$  be an isolated Pfaffian singularity, defined by the  $2m \times 2m$  Pfaffians of a skew-symmetric  $(2n + 1) \times (2n + 1)$  matrix  $(f_{ij})$  of formal power series. If  $\dim V < 4(n - m) + 7$ , then  $V$  is smoothable, and the smoothing may be globalized in the sense of (3.1.1).*

*Proof.* Left to the reader; there are obvious variants to the arguments given in the proof of (6.2).

**PROPOSITION 6.4.** *Let  $Y$  be a compact analytic space with one singularity, at  $P$ . If  $H^2(\theta_Y) = 0$ , then any deformation of  $P$  may be realized by a deformation of  $Y$ .*

*Proof.* Let  $\text{Def } Y$ ,  $\text{Def } P$  be corresponding (convergent) moduli spaces, [37, 38]. There is a natural morphism  $\Phi: \text{Def } Y \rightarrow \text{Def } P$ ; if  $\Phi$  is smooth (which can be verified formally), then deformations of  $P$  can be globalized. We claim  $\Phi$  is smooth if  $H^2(\theta_Y) = 0$ . For, let  $\{U_i\}$  be a Stein covering of  $Y$  with  $P \in U_1$ ,  $P \notin U_i (i > 1)$ ; let  $\{\tilde{U}_i\}$  be a lifted Stein cover of an infinitesimal deformation  $\tilde{Y}$  of  $Y$  over some Artin ring  $A$ .



Let  $A' \rightarrow A$  be a surjection of Artin rings with one-dimensional kernel. Let  $U'_i \rightarrow \bar{U}_i$  be a lifting over  $A'$  corresponding to a given further deformation of  $P$ . Lifting the  $\bar{U}_i$  arbitrarily to some  $U'_i (i > 1)$ , the deformations are isomorphic over  $\bar{U}_i \cap \bar{U}_j$ ; to patch to a global deformation of  $\bar{Y}$  (i.e. to get compatibility on triple overlaps), there is an obstruction in  $H^2(\theta_Y)$  (see [41], p. 31). The claim follows.

**THEOREM 6.5** *Let  $V$  be a two-dimensional cusp. Then there is a compact analytic surface  $Y$  which globalizes  $V$  and is non-singular elsewhere, for which  $H^2(\theta_Y) = 0$ .*

*Proof.* A construction of Inoue[16] gives a family of compact complex (non-singular) surfaces  $M$  (of class VII<sub>0</sub>) with the following properties:

- (a) the only irreducible curves on  $M$  are  $C_1, \dots, C_r, D_1, \dots, D_s$ , where  $C = \sum C_i$  and  $D = \sum D_i$  are the exceptional configurations of two "dual" cusps; further,  $C$  and  $D$  are disjoint
- (b) the canonical line bundle  $\omega_M = \mathcal{O}(-C - D)$
- (c)  $h^{p,q} = \dim H^q(\Omega^p)$  is zero, except for  $h^{0,0} = h^{2,2} = 1$ ;  $h^{0,1} = h^{2,1} = 1$ ;  $h^{1,1} = r + s$ .

Since any cusp configuration arises as such a  $C$  (see also [14]), and since the exceptional configuration uniquely determines the cusp[19], we may consider an  $M$  as yielding the cusp we started with. Let  $f: M \rightarrow Y$  be the result of blowing down  $C$  to a point (the cusp).  $Y$  will be the desired globalization, once we show  $H^2(\theta_Y) = 0$  (we also show  $H^2(\mathcal{O}_Y) = 0$ ).

Since  $f_*\mathcal{O}_M = \mathcal{O}_Y$ ,  $f_*\theta_M = \theta_Y$  the Leray spectral sequence yields

$$0 \rightarrow H^1(\mathcal{O}_Y) \rightarrow H^1(\mathcal{O}_M) \rightarrow H^0(R^1f_*\mathcal{O}_M) \rightarrow H^2(\mathcal{O}_Y) \rightarrow H^2(\mathcal{O}_M)$$

$$0 \rightarrow H^1(\theta_Y) \rightarrow H^1(\theta_M) \rightarrow H^0(R^1f_*\theta_M) \rightarrow H^2(\theta_Y) \rightarrow H^2(\theta_M).$$

A local computation gives  $H^0(R^1f_*\mathcal{O}_M) = H^1(\mathcal{O}_C) (= \mathbb{C})$ ; and  $H^0(R^1f_*\theta_M) = H^1(\theta_M \otimes \mathcal{O}_C)$  (this follows by tautness of cusps, which implies  $H^0(R^1f_*\theta_M) \cong \oplus H^1(N_{C_i})$ ). Thus,  $h^2(\mathcal{O}_Y) = h^2(\mathcal{O}_M(-C))$ ,  $h^2(\theta_Y) = h^2(\theta_M(-C))$ . But

$$(6.5.1) \quad h^2(\mathcal{O}_M(-C)) = h^0(\omega_M(C)) = h^0(\mathcal{O}(-D)) = 0$$

$$h^2(\theta_M(-C)) = h^0(\Omega_M^1 \otimes \omega_M(C)) = h^0(\Omega_M^1(-D)) \leq h^0(\Omega_M^1) = 0.$$

**PROPOSITION 6.6.** *A quotient singularity  $V = \mathbb{C}^2/G$  may be globalized to a projective  $Y$  with  $H^2(\theta_Y) = H^2(\mathcal{O}_Y) = 0$ .*

*Proof.* The action of  $G \subset GL(2)$  on  $\mathbb{C}^2$  is free off  $(0, 0)$ , and extends to an action on  $\mathbb{P}^2$ .  $G$  may have isotropy along the curve at  $\infty$ ; nonetheless, form  $\pi: \mathbb{P}^2 \rightarrow \mathbb{P}^2/G = Y'$ .  $Y'$  is a compactification of  $V$ , with isolated quotient singularities at  $\infty$ . The argument of ([32], §2) indicates

$$(6.6.1) \quad (\pi_*\theta_{\mathbb{P}^2})^G \subset \theta_{Y'}$$

$$(6.6.2) \quad (\pi_*\theta_{\mathbb{P}^2})^G \text{ is a direct summand of } \pi_*\theta_{\mathbb{P}^2}.$$

Since  $H^2(\theta_{\mathbb{P}^2}) = 0$  and  $\pi$  is finite, (6.6.2) gives  $H^2(Y', (\pi_*\theta_{\mathbb{P}^2})^G) = 0$ ; since the cokernel of (6.6.1) is supported along the curve at  $\infty$ , it follows that  $H^2(\theta_{Y'}) = 0$ . Letting  $Y \rightarrow Y'$  be the minimal resolution of the singular points at  $\infty$ , we immediately deduce  $H^2(\theta_Y) = H^2(\mathcal{O}_Y) = 0$  (use Leray).

**Remark (6.7).** More generally, one can prove (6.6) for a rational  $V$  with  $C^*$ -action, all of whose deformations have weight  $\leq 0$ ; however, we omit the proof.

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Since the initial draft of this paper, E. Looijenga has made significant progress in understanding the Milnor fibre  $F$  of a cusp singularity; in particular, using the Inoue globalization (6.5), he shows  $F$  is a rational surface minus a divisor, and  $b_1(F) = 0$ .

**Note added in proof.** (1) The definition of Milnor fibre in §1 should be  $F = \tilde{B}_r \cap V_r$ , with  $\tilde{F} = F \cup \partial F$ ; then consider  $H^i(\tilde{F}, \partial F)$ .  $F$  and  $\tilde{F}$  have the same cohomology, but are not diffeomorphic ( $\tilde{F}$  is compact). (2) The work of Looijenga on cusps appears in a preprint, Rational surfaces with anti-cononical cycle.—I. Nakamura has also used Inoue surfaces to study cusps (to appear in *Math. Ann.*). We have recently shown that a smoothing component of a cusp has dimension  $\beta = r + 10 - m$ .

#### REFERENCES

1. A. ANDREOTTI and T. FRANKEL: The Lefschetz theorem on hyperplane sections. *Annals Math.* **69** (1959), 713–717.
2. B. BENNETT and S. S.-T. YAU: *Milnor number and invariants of strongly pseudoconvex manifolds*, preprint (1979).
3. D. BUCHSBAUM and D. EISENBUD: Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3. *Am. J. Math.* **99** (1977), 447–485.
4. D. BUCHSBAUM and D. EISENBUD: to appear.
5. H. CARTAN and S. EILENBERG: *Homological Algebra*. Princeton University Press, Princeton (1956).
6. P. DELIGNE: Groupes de monodromie en géométrie algébrique. (SGA 7, I), Exp. 1, *Lecture Notes in Mathematics* **288**, Springer-Verlag.
7. P. DELIGNE and N. KATZ: Groupes de monodromie en géométrie algébrique. (SGA 7, II), Exp. 10, *Lecture Notes in Mathematics* **340**, Springer-Verlag.
8. I. DOLGACHEV: *Weighted Projective Spaces*, to appear.
9. A. DURFEE: The signature of smoothings of complex surface singularities. *Math. Ann.* **232** (1978), 85–98.
10. P. GRIFFITHS and J. HARRIS: *Principles of Algebraic Geometry*. Wiley, New York (1978).
11. G.-M. GREUEL: Dualität in der lokalen Kohomologie isolierter Singularitäten. to appear.
12. H. HAMM: Lokale topologische Eigenschaften komplexer Räume. *Math. Ann.* **191** (1971), 235–252.
13. F. HIRZBRUCH: *Topological Methods in Algebraic Geometry*. Springer-Verlag, New York (1966).
14. F. HIRZBRUCH: Hilbert modular surfaces. *L'Enseignement Math.* **19** (1973), 183–281.
15. M. HOCHSTER: The Zariski–Lipman conjecture in the graded case. *J. Alg.* **47** (1977), 411–424.
16. M. INOUE: New surfaces with no meromorphic functions, II. In *Complex Analysis and Algebraic Geometry* (Edited by W. Baily and T. Shioda). Cambridge University Press (1977).
17. K. KODAIRA: On the structure of compact complex analytic surfaces, I. *Am. J. Math.* **86** (1964), 751–798.
18. D. LAKSOV: Deformation of determinantal schemes. *Compositio Math.* **30** (1975), 273–292.
19. H. LAUFER: Taut two-dimensional singularities. *Math. Ann.* **205** (1973), 131–164.
20. H. LAUFER: On  $\mu$  for surface singularities. *Symp. in Pure Math.* **30** (1977), 45–49.
21. LÊ DŨNG TRÁNG: Some remarks on relative monodromy. In *Proc. of the Nordic Summer School*. Sijthoff and Noordhoff, Oslo (1976).
22. J. LIPMAN: Double point resolutions of deformations of rational singularities. *Compositio Math.* **38** (1979), 37–43.
23. J. MILNOR: *Singular Points of Complex Hypersurfaces*. Ann. Math. Studies **91**. Princeton University Press, Princeton (1968).
24. D. MUMFORD: An algebraic surface with  $K$  ample,  $(K^2) = 9$ ,  $p_g = q = 0$ . *Am. J. Math.* **101** (1979), 233–244.
25. W. NEUMANN: An invariant of plumbed homology spheres. *Proc. Intern. Topol. Symp. Siegen* (1979), to appear.
26. H. PINKHAM: Deformations of algebraic varieties with  $G_m$  action. *Astérisque* **20** (1974), 1–131.
27. H. PINKHAM: Deformations of normal singularities with  $C^*$ -action. *Math. Ann.* **232** (1978), 65–84.
28. O. RIEMENSCHNEIDER: Deformationen von Quotientensingularitäten (nach zyklischen Gruppen). *Math. Ann.* **209** (1977), 377–400.
29. O. RIEMENSCHNEIDER and K. BEHNKE: Infinitesimale Deformationen von Diedersingularitäten. *Manus. Math.* **20** (1977), 377–400.
30. M. SCHAPS: Deformations of Cohen–Macaulay schemes of codimension 2 and non-singular deformations of space curves. *Am. J. Math.* **99** (1977), 669–685.
31. M. SCHLESSINGER: Rigidity of quotient singularities. *Invent. Math.* **14** (1971), 17–26.
32. J. STEENBRINK: Intersection form for quasi-homogeneous singularities. *Compositio Math.* **34** (1977), 211–223.
33. B. TEISSIER: The hunting of invariants in the geometry of discriminants. In *Proc. of the Nordic Summer School*. Sijthoff and Noordhoff, Oslo (1976).
34. J. WAHL: Simultaneous resolution of rational singularities. *Compositio Math.* **38** (1979), 43–54.
35. J. WAHL: Elliptic deformations of minimally elliptic singularities. Preprint (1979). *Math. Ann.*, to appear.
36. S. S.-T. YAU: The signature of Milnor fibres and duality theorem for strongly pseudoconvex manifolds. *Invent. Math.* **46** (1978), 81–97.

37. H. GRAUERT: Deformationen isolierter Singularitäten. *Inv. Math.* **15** (1972), 171–198.  
 38. H. GRAUERT: Der Satz von Kuranishi für kompakte komplexe Räume. *Inv. Math.* **25** (1974), 107–142.  
 39. O. LAUDAL: *Formal Moduli of Algebraic Structures. Lecture Notes in Mathematics* **754**, Springer-Verlag (1979).  
 40. H. LAUFER: On minimally elliptic singularities. *Am. J. Math.* **99** (1977), 1257–1295.  
 41. M. SCHLESSINGER: Infinitesimal deformations of singularities. Thesis, Harvard (1964).

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#### APPENDIX: DUALIZING AND SPECIALIZING

THEOREM A.1. Let  $R$  be a local ring,  $t \in R$  a non-0 divisor,  $\bar{R} = R/tR$ . Suppose  $M$  is a finite  $R$ -module, locally free off the closed point, and  $\bar{M} = M/tM$ . Denote dual by  $*$ , and length by  $l$ .

(a) For all integers  $n \geq 0$ ,

$$l(\bar{M}^*/\bar{M}^*) = \sum_{i=1}^n (-1)^{i+1} l(\text{Ext}_R^i(M, \bar{R})) + (-1)^{n+2} \delta_n,$$

where  $0 \leq \delta_n \leq l(\text{Ext}_R^{n+1}(M, \bar{R}))$ .

(b)  $\delta_n = 0$  iff  $\text{Ext}_R^{n+1}(M, R) = 0$ .

(c) If  $T(M) = \{m \in M \mid tm = 0\}$ , then

$$l(\text{Ext}_R^i(M, \bar{R})) = l(\text{Ext}_R^i(\bar{M}, \bar{R})) + l(\text{Ext}_R^{i-1}(T(M), \bar{R})).$$

*Proof.* Applying  $\text{Hom}_R(M, \quad)$  to

$$0 \rightarrow R \xrightarrow{t} R \rightarrow \bar{R} \rightarrow 0$$

yields

$$\begin{aligned} 0 \rightarrow M^* \rightarrow M^* \rightarrow \text{Hom}_R(M, \bar{R}) \rightarrow \text{Ext}_R^1(M, R) \rightarrow \text{Ext}_R^1(M, R) \\ \rightarrow \text{Ext}_R^1(M, \bar{R}) \rightarrow \text{Ext}_R^2(M, R) \rightarrow \dots \end{aligned}$$

As  $M$  is locally free off the closed point,  $\text{Ext}_R^i(M, R)$  has finite length if  $i > 0$ ; writing

$$0 \rightarrow F_i \rightarrow \text{Ext}_R^i(M, R) \rightarrow \text{Ext}_R^i(M, R) \rightarrow G_i \rightarrow 0,$$

we have  $l(F_i) = l(G_i)$ ;  $i > 0$ . Using

$$0 \rightarrow G_i \rightarrow \text{Ext}_R^i(M, \bar{R}) \rightarrow F_{i+1} \rightarrow 0,$$

we deduce

$$(*) \quad l(F_i) = l(\text{Ext}_R^i(M, \bar{R})) - l(F_{i+1}).$$

Note the isomorphism of  $\bar{R}$ -modules

$$\text{Hom}_R(M, \bar{R}) \xrightarrow{\sim} \text{Hom}_{\bar{R}}(\bar{M}, \bar{R}) = \bar{M}^*.$$

Therefore, from the long exact sequence,

$$l(\bar{M}^*/\bar{M}^*) = l(F_1).$$

Iterating (\*) then gives (a) above, with  $\delta_n = l(F_{n+1})$ . For (b), note  $F_{n+1} = 0$  implies multiplication by  $t$  on  $\text{Ext}_{\bar{R}}^{n+1}(M, R)$  is injective; as this module is supported at the maximal ideal of  $R$ , it must then be 0. The converse of (b) is also clear.

Finally, recall the change of rings spectral sequence ([5], p. 348)

$$E_2^{p,q} = \text{Ext}_{\bar{R}}^q(\text{Tor}_p^R(\bar{R}, M), \bar{R}) \Rightarrow \text{Ext}_{\bar{R}}^n(M, \bar{R}).$$

But

$$\text{Tor}_p^R(\bar{R}, M) = 0, \quad p \geq 2$$

$$\text{Tor}_1^R(\bar{R}, M) = \text{Ker}(M \xrightarrow{t} M) = T(M).$$

Therefore, the spectral sequence degenerates, yielding (c),

COROLLARY A. 2. *Assuming further depth  $\bar{R} \geq 2$ , we have*

$$l(\text{Ext}_{\bar{R}}^1(\bar{M}, \bar{R})) - l(\text{Ext}_{\bar{R}}^2(\bar{M}, \bar{R})) \leq l(\bar{M}^*/\overline{M^*}) \leq l(\text{Ext}_{\bar{R}}^1(\bar{M}, \bar{R})).$$

*Proof.* Use (a) with  $n = 1$ , plus (c); observe the depth condition implies  $\text{Ext}_{\bar{R}}^i(\bar{N}, \bar{R}) = 0$ ,  $i = 0, 1$ , for any  $\bar{R}$ -module  $\bar{N}$  of finite length.